Int J Theor Phys (2010) 49: 2002–2015
DOI 10.1007/s10773-010-0384-4

Geometric Control for Unified Entangling Quantum Gate with High-Fidelity in Electric Circuit

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Received: 17 March 2010 / Accepted: 12 May 2010 / Published online: 26 May 2010

Abstract Implements for geometric quantum gates are analyzed in the electric circuit. We find that, by operating the difference of geometric phase between eigenstates of Hamiltonian for single-particle system and two-particle system respectively, one may perfectly preserve the messages for single-particle gate as well as entangling geometric two-particle gate.

Keywords Geometric phase · Fidelity · Geometric quantum gate · Entangling geometric two-particle gate

1 Introduction

Geometric phases have been attracting an increasing interest in understanding and implementing quantum computation in real physical systems [1]. Geometric quantum computation demands that logical gates in computing are realized by using geometric phase shifts, so that it may have the built-in fault-tolerant advantage due to the fact that the geometric phases depend only on some global geometric features.

Theoretically, a pure geometric quantum gate can be achieved based only on adiabatic geometric phases [1]. However, it is difficult to experimentally realize the quantum computation with adiabatic evolution because the long operation time is required, especially for solid-state systems whose decoherence time is very short. To solve this problem, Aharonov and Anandan phase (A-A phase) was suggested to realize the geometric quantum gates, which allow for fastening gate-operation time and have an intrinsically geometric feature [2].

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Decoherence is the most important limiting factor for quantum computation because its effect is that quantum superpositions decay into statistical mixtures [3, 4]. It may be better, therefore, to construct geometric quantum gates by using the nonadiabatic geometric phase [5, 6] since this allows for shortening gate times. For a nonadiabatic cyclic evolution, the total phase between the final and initial states is sum of the geometric and dynamical phases. In the some methods of geometric quantum computation, it is necessary to remove the dynamical component, such as by using dark states [7] and by rotating operations in so-called single-loop and multi-loop schemes [8, 9]. The experimental errors are, obviously, increased because of the operational process. More worryingly, the dynamic phase accumulated in the gate operation is possibly nonzero and cannot be eliminated. Therefore, it may be better to realize nonadiabatic geometric quantum computation by using varying parameters in the Hamiltonian, where the dynamical and geometric phases are implemented separately without the usual operational process [5, 10, 11].

In a really closed system, a useful way to remove the adiabatic constraint in quantum computation is the theory of dynamical invariant to treat time-dependent Hamiltonian [12–14]. Indeed, dynamically invariant theory were recently used in a proposal of an interferometric experiment to measure nonadiabatic geometric phase in cavity quantum electrodynamics [10].

Fidelity is a degree of closeness to the original state of information in the transmission, which is of fundamental importance in information science as well as in the quantum optics. It has been adopted broadly as an important physical parameter in quantum communication and quantum computation [15, 16]. Fidelity can measure the performance of quantum teleportation and describe the similarity between the input state and the output state. In order to obtain a better output result, the efforts of increasing the fidelity of quantum teleportation in continuous-variable systems have greatly been made [17–19]. In order to perfectly preserve the initial quantum state in the geometric quantum computation, therefore, it is interesting to study the relations between the fidelity and parameters of electric circuit and between fidelity and geometric phases.

2 Jaynes-Cummings Model and Electric Circuit

It is known that the superconducting qubits are promising building blocks for the realization of a quantum computer. Therefore, it may be important to investigate geometric gates in system of superconducting qubits coupling with a transmission line resonator [20–22].

The qubits in the circuit QED architecture are split-junction Cooper pair boxes. These devices can be modeled as two-level systems so as to be regarded as artificial atoms with large dipole moments, and in circuit QED they are coupled to microwave frequency photons in a quasi-one-dimensional transmission line cavity (a coplanar waveguide resonator) by an electric dipole interaction. For the qubit and transmission line cavity, the Hamiltonian is the well-known Jaynes-Cummings Hamiltonian (JCM),

$$H(t)=\frac{1}{2}\omega_J\sigma_z+\omega_F\left(a^+a+\frac{1}{2}\right)+g(t)a^+\sigma_-+g^*(t)a\sigma_+, \tag{1}$$

where a^+ and a denote the photon creation and annihilation operators satisfying the commutation relation $[a,a^+]=1$, $\sigma_\pm$ and σ_z are Pauli matrices acting on the states of the two-level system, ω_F is the cavity resonance frequency, $\omega_J=E_J$ is the qubit transition frequency. While $g(t)=g_0\exp i(\omega t-\frac{\pi}{2})$ is the coupling constant for the interaction between the two-level state and cavity, $g^*(t)$ is complex conjugation of $g(t)$ and $g_0=\frac{ec_J}{2(c_g+c_J)}\sqrt{\frac{\omega_F}{2\hbar c_F}}$, where

c_F is a parameter of Josephson junction [22], c_g and c_J are gate and tunnel junction capacitances respectively.

3 Invariant Operator and Solution of Schördinger Equation

For a closed quantum system, a dynamical invariant operator $I(t)$ satisfies

$$\frac{\partial I(t)}{\partial t} = i[I(t), H(t)], \tag{2}$$

where $H(t)$ is the Hamiltonian of the system and $I(t)$ is a Hermitian invariant operator with a member of a complete set of commuting observables [12–14]. Therefore, there exists a set of simultaneous eigenfunctions $|\lambda_n, a, t\rangle$ satisfying

$$I(t)|\lambda_n, a, t\rangle = \lambda_n|\lambda_n, a; t\rangle, \tag{3}$$

$$\langle\lambda_m, b; t|\lambda_n, a, t\rangle = \delta_{mn}\delta_{ba}, \tag{4}$$

where λ_n is an eigenvalue of the invariant operator $I(t)$, while a and b are degenerate labels. $|\lambda_n, a; t\rangle$ are also eigenstates of the Hamiltonian $H(t)$. Therefore, an exact solution of Schrödinger equation with the Hamiltonian $H(t)$ can be expressed as

$$|\psi(t)\rangle = \sum_{n,a} c_{n,a}\mathcal{P}(e^{i\chi_{n,a}})|\lambda_n, a, t\rangle, \tag{5}$$

where c_n does not depend on the involving time and $\mathcal{P}$ stands for the time-ordering operator. The phases $\chi_{n,a}$ are determined by [12–14]

$$\chi_{n,a} = \int_0^t dt\langle\lambda_n, a, t|i\frac{\partial}{\partial t} - H(t)|\lambda_n, a, t\rangle, \tag{6}$$

where the first term is geometric phase and the second term is dynamical phase.

4 Single-Particle Geometric Quantum Gate

It is known that the supersymmetric structure was found to embed in the JCM [23]. Therefore it is interesting to physically implement a universal set of quantum gates based on the dynamically supersymmetric invariant.

By defining the tensor operators and coupling constant [23],

$$V = \sigma_+\sigma_- + a^+a, \qquad M = \sigma_+\sigma_- - a^+a - 1, \tag{7}$$

$$Q_+ = Q_-^+ = \frac{1}{\sqrt{2}}\sigma_-a^+, \qquad G(t) = \sqrt{2}g(t), \tag{8}$$

the Hamiltonian (1) is rewritten as

$$H(t) = \frac{1}{2}(\omega_J + \omega_F)V + \frac{1}{2}(\omega_J - \omega_F)M + G(t)Q_+ + G^*(t)Q_-, \tag{9}$$

which can be easily recognized as an element of the superalgebra associated with the unitary supergroup $u(1,1)$. The tensor operators satisfy the commutation relations,

$$\{Q_\epsilon, Q_\eta\} = \frac{1}{2}V\delta_{\epsilon,-\eta}, \qquad [V, M] = 0 = [V, Q_\epsilon], \tag{10}$$

$$[M, Q_\epsilon] = -2\epsilon Q_\epsilon \quad (\epsilon, \eta = \pm), \tag{11}$$

where V is a Casimir operator. Equations (10) and (11) imply that the set of operators $\{V, M, Q_+, Q_-\}$ generates a dynamically closed superalgebra.

In order to compute the geometric phases, we require to get the eigenstates of the invariant operator. In accordance with the closed superalgebra theory, the invariant operator $I(t)$ is of the general form, such as

$$\begin{aligned} I(t) = \frac{1}{2}(\Omega_1(t) + \Omega_2(t))V + \frac{1}{2}(\Omega_1(t) - \Omega_2(t))M \\ + \xi(t)Q_+ + \xi^*(t)Q_-, \end{aligned} \tag{12}$$

where $\Omega_1(t), \Omega_2(t), \xi(t)$ and $\xi^*(t)$ are different from $\omega_J, \omega_F, G(t)$ and $G^*(t)$ respectively and will be determined by (1) and the closed superalgebra theory.

Inserting (6) and (12) into (1) and using (10)–(11), we find that $\Omega_1(t)$ and $\Omega_2(t)$ are arbitrary constant because of the relations $\dot{\Omega}_2(t) = 0$ and $\dot{\Omega}_1(t) = 0$, and $\xi(t) = (\sqrt{2}(\Omega_1 - \Omega_2)g_0/(\omega - \omega_J + \omega_F))\exp i(\omega t + \frac{\pi}{2})$. The eigenvalues of operator $I(t)$ can be obtained by $\lambda_n^\pm = \Omega(n+1) \pm \frac{1}{2}\sqrt{(\Omega_1 - \Omega_2)^2 + 2(n+1)|\xi|^2}$, where n is photon number. It is noted that the eigenvalues are independent of time. The corresponding eigenstates are expressed as

$$|\lambda_n^+, t\rangle = \begin{pmatrix} \cos(\theta_n/2)|n\rangle \\ e^{i\beta}\sin(\theta_n/2)|n+1\rangle \end{pmatrix}, \tag{13}$$

$$|\lambda_n^-, t\rangle = \begin{pmatrix} -\sin(\theta_n/2)|n\rangle \\ e^{i\beta}\cos(\theta_n/2)|n+1\rangle \end{pmatrix}, \tag{14}$$

where $e^{i\beta} = \xi(t)/|\xi(t)|$, $|n\rangle$ with $\beta = \omega t + \pi/2$ is a photon number state and $\theta_n = 2\arctan[\sqrt{1+G_n^2} - G_n]$ with $G_n = (\omega - \omega_J + \omega_F)/2\sqrt{n+1}g_0$. The JCM describes generally a nonresonant interaction between a two-level system with lower state $|-\rangle$, upper state $|+\rangle$, and a harmonic oscillator denoted by the photon number state $|n\rangle$.

Using (5) and (13)–(14), we know that the geometric phases are given by

$$\gamma_{n+}^g = -\pi(1 - \cos\theta_n), \qquad \gamma_{n-}^g = -\pi(1 + \cos\theta_n), \tag{15}$$

and the corresponding dynamical phases are expressed as

$$\gamma_{n+}^d = \frac{2\pi}{\omega}\left[\frac{1}{2}(\omega_J - \omega_F)\cos\theta_n + \omega_F(n+1) + g_0\sqrt{n+1}\sin\theta_n\right], \tag{16}$$

$$\gamma_{n-}^d = \frac{2\pi}{\omega}\left[-\frac{1}{2}(\omega_J - \omega_F)\cos\theta_n + \omega_F(n+1) - \sqrt{n+1}g_0\sin\theta_n\right]. \tag{17}$$

It is interesting to note that, for the photon number $n = 0$, the phases are different from zero, which mean that the vacuum field introduces a correction in the geometric phases and dynamical phases [14].

We set

$$\frac{1}{2}(\omega_J - \omega_F)\cos\theta_n + \sqrt{n+1}g_0 \sin\theta_n = 2k\omega \quad (k = 0, 1, 2, \ldots). \tag{18}$$

Under the condition, according to (4), the wave function for single-qubit system may be expressed as

$$|\psi(t)\rangle = e^{i\gamma_{n+}^g}|\lambda_n^+, t\rangle + e^{i\gamma_{n-}^g}|\lambda_n^-, t\rangle, \tag{19}$$

where the overall phase factor $\exp[i\frac{2\pi\omega_F}{\omega}(n+1)]$ is dropped off because it is not important for the quantum computation. According to (19), under the computational basis $\{|0\rangle = \binom{|n\rangle}{0}$, $|1\rangle = \binom{0}{|n+1\rangle}\}$ where the qubit includes information about the state of the photon, the geometric quantum gate for single-particle system in the cyclic evolution can be written as

$$u(\gamma_{n+}^g, \gamma_{n-}^g, \theta_n) = \begin{pmatrix} X & R \\ R & Y \end{pmatrix}, \tag{20}$$

where $X = e^{i\gamma_{n+}^g}\cos^2\frac{\theta_n}{2} + e^{i\gamma_{n-}^g}\sin^2\frac{\theta_n}{2}$, $Y = -e^{i\gamma_{n+}^g}\sin^2\frac{\theta_n}{2} - e^{i\gamma_{n-}^g}\cos^2\frac{\theta_n}{2}$ and $R = \frac{1}{2}i\sin\theta_n(e^{i\gamma_{n+}^g} - e^{i\gamma_{n-}^g})$.

5 Two-Particle Geometric Quantum Gate

For superconducting two-qubit system in JCM, in order to simplify our computation but without loss of generality, we only consider the Casimir interaction between the control and target qubits. The total Hamiltonian is

$$\begin{aligned} H_{12}(t) = \frac{1}{2}(\omega_J + \omega_F)V_1 + \frac{1}{2}(\omega_J - \omega_F)M_1 \\ + G(t)Q_{1+} + G^*(t)Q_{1-} + \lambda V_1 V_2, \end{aligned} \tag{21}$$

where λ is the strength of the interaction between two qubits. Similarly to the one-qubit system, the invariant operator for the two-qubit system may be expressed as

$$\begin{aligned} I_{12}(t) = \frac{1}{2}(\Omega_1 + \Omega_2)V_1 + \frac{1}{2}(\Omega_1 - \Omega_2)M_1 \\ + \xi(t)Q_{1+} + \xi^*(t)Q_{1-} + \Lambda(t)V_1 V_2. \end{aligned} \tag{22}$$

Substituting (21) and (22) into (1), we find that Ω_1, Ω_2 and Λ are constant, while $\xi(t)$ is the same with the one in the single-qubit system.

Under the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, the invariant operator $I_{12}(t)$ may be rewritten as

$$I_{12}(t) = \begin{pmatrix} I'_{11} & 0 & \sqrt{\frac{n+1}{2}}\xi^* & 0 \\ 0 & I'_{11} & 0 & \sqrt{\frac{n+1}{2}}\xi^* \\ \sqrt{\frac{n+1}{2}}\xi & 0 & I'_{22} & 0 \\ 0 & \sqrt{\frac{n+1}{2}}\xi & 0 & I'_{22} \end{pmatrix}, \tag{23}$$

where $I'_{11} = \Omega_2(n+1/2) + \Omega_1/2 + \Lambda(n+1)^2$ and $I'_{22} = \Omega_2(n+3/2) - \Omega_1/2 + \Lambda(n+1)^2$. The eigenvalues of operator $I_2(t)$ are degenerate with $\lambda_n^{\pm} = \Omega_2(n+1) + \Lambda(n+$

$1)^2 \pm \frac{1}{2}\sqrt{(\Omega_1 - \Omega_2)^2 + 2(n+1)|\xi|^2}$ and the corresponding eigenstates are $|\lambda_n^+, 1, t\rangle = \cos(\theta_n/2)|00\rangle + e^{i\beta}\sin(\theta_n/2)|10\rangle)$, $|\lambda_n^+, 2, t\rangle = \cos(\theta_n/2)|01\rangle + e^{i\beta}\sin(\theta_n/2)|11\rangle)$, $|\lambda_n^-, 1, t\rangle = -\sin(\theta_n/2)|00\rangle + e^{i\beta}\cos(\theta_n/2)|10\rangle)$ and $|\lambda_n^-, 2, t\rangle = -\sin(\theta_n/2)|01\rangle + e^{i\beta}\cos(\theta_n/2)|11\rangle)$, respectively. Using these eigenfuctions, we find that the geometric phases are

$$\gamma_{n+}^g(1) = \gamma_{n+}^g(2) = \gamma_{n+}^g, \qquad \gamma_{n-}^g(1) = \gamma_{n-}^g(2) = \gamma_{n-}^g, \tag{24}$$

and the corresponding dynamical phases are

$$\gamma_{n+}^d(1) = \gamma_{n+}^d(2) = \frac{2\pi}{\omega}\bigg[\omega_F(n+1) + \lambda(n+1)^2 + \frac{1}{2}(\omega_J - \omega_F)\cos\theta_n + g_0\sqrt{n+1}\sin\theta_n\bigg], \tag{25}$$

$$\gamma_{n-}^d(1) = \gamma_{n-}^d(2) = \frac{2\pi}{\omega}\bigg[\omega_F(n+1) + \lambda(n+1)^2 - \frac{1}{2}(\omega_J - \omega_F)\cos\theta_n - g_0\sqrt{n+1}\sin\theta_n\bigg]. \tag{26}$$

Under the condition (18), the wave functions may be expressed by

$$\Psi(t) = c_1 e^{i\gamma_{n+}^g}|\lambda_n^+, 1, t\rangle + c_2 e^{i\gamma_{n+}^g}|\lambda_n^+, 2, t\rangle + c_3 e^{i\gamma_{n-}^g}|\lambda_n^-, 1, t\rangle + c_4 e^{i\gamma_{n-}^g}|\lambda_n^-, 2, t\rangle, \tag{27}$$

where the overall phase factor is dropped off.

Thus the geometric quantum gate for two-particle system under the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ may be expressed as

$$U(\gamma_{n+}^g, \gamma_{n-}^g, \theta_n) = \begin{pmatrix} X & 0 & R & 0 \\ 0 & X & 0 & R \\ R & 0 & Y & 0 \\ 0 & R & 0 & Y \end{pmatrix}, \tag{28}$$

which is an entangling universal quantum gates based entirely on purely geometric operations (holonomies).

6 Fidelity of Single-Particle Gate

It is well-known that fidelity measures the overlap between the initial and time-developed state vectors [24–26]. Therefore, one can know that the performance of quantum information and the similarity between the input and the output states. For an initially pure state $|\psi(0)\rangle$, the fidelity is, in fact, a probability to find the initial state in the output state at a later time. Thus the properties of quantum information through some quantum channels are usually quantified by the fidelity. The fidelity was defined by Schumacher as

$$F(t_1, t_2) = \Big[\mathrm{Tr}\Big(\sqrt{\rho(t_1)}\rho(t_2)\sqrt{\rho(t_1)}\Big)^{\frac{1}{2}}\Big]^2, \tag{29}$$

where $\rho(t_1) = |\Psi(t_1)\rangle\langle\Psi(t_1)|$ and $\rho(t_2) = |\Psi(t_2)\rangle\langle\Psi(t_2)|$ are the density operators corresponding to the initial and final pure states, respectively. If the initial and the final states

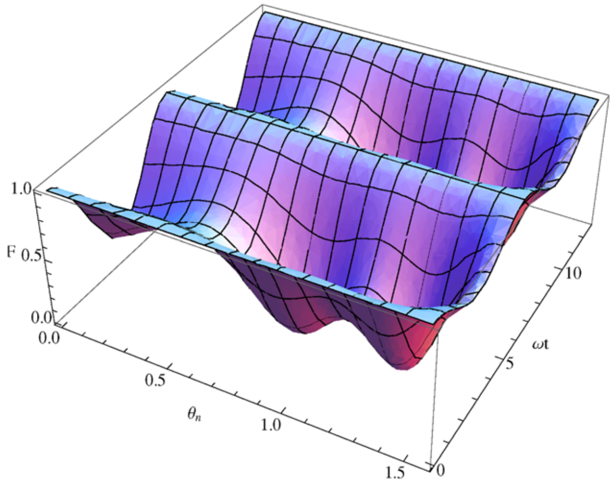

Fig. 1 Fidelity for single-particle gate as functions of θ_n and ωt, where $c_1 = \cos(\pi/3)$, $c_2 = \sin(\pi/3)$. The result show that, when $\omega t = 2\pi, 4\pi$, $F = 1$ does not depend on the θ_n

are orthogonal each other, then $F(t_1, t_2) = 0$, which indicates that the quantum information is totally lost in the quantum computation. If the initial and final states are coincided, then $F(t_1, t_2) = 1$, which implies that the quantum information is protected in the transmission. If $0 < F(t_1, t_2) < 1$, certain distortion exists in the transmission process of information.

For the evolution of pure states, based on (29), we have

$$F(t_1, t_2) = |\langle \Psi(t_1)|\Psi(t_2)\rangle|^2. \tag{30}$$

In terms of (13), (14) and (19), at the initial time $t = 0$, the state vector of the one-qubit system may be written as

$$|\psi(0)\rangle = \begin{pmatrix} (c_1 e^{i\gamma_{n+}^g}\cos(\theta_n/2) - c_2 e^{i\gamma_{n-}^g}\sin(\theta_n/2))|n\rangle \\ i(c_1 e^{i\gamma_{n+}^g}\sin(\theta_n/2) + c_2 e^{i\gamma_{n-}^g}\cos(\theta_n/2))|n+1\rangle \end{pmatrix}, \tag{31}$$

at the evolving time $t > 0$, the state vector of the one-qubit system may be written as

$$|\psi(t)\rangle = \begin{pmatrix} (c_1 e^{i\gamma_{n+}^g}\cos(\theta_n/2) - c_2 e^{i\gamma_{n-}^g}\sin(\theta_n/2))|n\rangle \\ ie^{i\omega t}(c_1 e^{i\gamma_{n+}^g}\sin(\theta_n/2) + c_2 e^{i\gamma_{n-}^g}\cos(\theta_n/2))|n+1\rangle \end{pmatrix}, \tag{32}$$

where c_1 and c_2 are determined by the initial condition. The state vector $|\psi(t)\rangle$ is constructed based on the geometric phase shifts and therefore called the geometric qubit state.

Substituting (32) and (31) into (30), we find that the fidelity may be expressed as

$$\begin{aligned} F_1(t) = \big| & c_1c_1^*\cos^2(\theta_n/2) + c_2c_2^*\sin^2(\theta_n/2) - c_1c_2^*\cos(\theta_n/2)\sin(\theta_n/2)e^{i\gamma} \\ & - c_2c_1^*\cos(\theta_n/2)\sin(\theta_n/2)e^{-i\gamma} + e^{i\omega t}(c_1c_1^*\sin^2(\theta_n/2) + c_2c_2^*\cos^2(\theta_n/2) \\ & + c_1c_2^*\cos(\theta_n/2)\sin(\theta_n/2)e^{i\gamma} + c_2c_1^*\cos(\theta_n/2)\sin(\theta_n/2)e^{-i\gamma})\big|^2, \end{aligned} \tag{33}$$

where $\gamma = \gamma_{n+}^g - \gamma_{n-}^g$ is difference of the geometric phases and depends on the θ_n. It is obvious that the fidelity is functions of the θ_n and ωt, while $\theta_n = 2\arctan[\sqrt{1+G_n^2} - G_n]$ with $G_n = (\omega - \omega_J + \omega_F)/2\sqrt{n+1}g_0$.

The single-particle fidelity $F_1(t)$ as functions of θ_n and ωt is shown at Fig. 1. We find that the fidelity oscillates in terms of both the θ_n and ωt. When $\omega t = 2n\pi\,(n = 0, 1, 2, 3, \ldots)$, especially, the fidelity is equal to one and not relative to the θ_n. Therefore the initial messages may be perfectly preserved if one choose the output state at the evolving time $t = 2n\pi/\omega (n = 1, 2, 3, \ldots)$.

7 Fidelity of Two-Particle Gate

The entangling universal geometric quantum gates is important for the geometric quantum computation. Therefore, it is interesting to study its fidelity. The fidelity of two qubits may be expressed as

$$\begin{aligned}F_2(t) = \big| & c_1 c_1^* \cos^2(\theta_n/2) + c_3 c_3^* \sin^2(\theta_n/2) - c_1 c_3^* \cos(\theta_n/2)\sin(\theta_n/2)e^{i\gamma} \\ & - c_3 c_1^* \cos(\theta_n/2)\sin(\theta_n/2)e^{-i\gamma} \\ & + e^{i\omega t}(c_1 c_1^* \sin^2(\theta_n/2) + c_3 c_3^* \cos^2(\theta_n/2) + c_1 c_3^* \cos(\theta_n/2)\sin(\theta_n/2)e^{i\gamma} \\ & + c_3 c_1^* \cos(\theta_n/2)\sin(\theta_n/2)e^{-i\gamma}) \\ & + c_2 c_2^* \cos^2(\theta_n/2) + c_4 c_4^* \sin^2(\theta_n/2) - c_2 c_4^* \cos(\theta_n/2)\sin(\theta_n/2)e^{i\gamma} \\ & - c_4 c_2^* \cos(\theta_n/2)\sin(\theta_n/2)e^{-i\gamma} \\ & + e^{i\omega t}(c_2 c_2^* \sin^2(\theta_n/2) + c_4 c_4^* \cos^2(\theta_n/2) + c_2 c_4^* \cos(\theta_n/2)\sin(\theta_n/2)e^{i\gamma} \\ & + c_4 c_2^* \cos(\theta_n/2)\sin(\theta_n/2)e^{-i\gamma})\big|^2, \end{aligned} \tag{34}$$

where $\gamma = \gamma_{n+}^g - \gamma_{n-}^g$ is a function of θ_n. Thus the fidelity is only functions of both the θ_n and ωt.

The two-particle fidelity as functions of θ_n and ωt is shown at Fig. 2. We see that the fidelity oscillates with the θ_n and ωt respectively. it is interesting in noting that, differently from the single-particle fidelity, the maximal values $F_2 = 1$ have two group at the $\theta_n = \pi/3$ or $\omega t = 2n\pi (n = 0, 1, 2, 3, \ldots)$ respectively. Especially, they are independent each other, which provide a wide choice for the experimenter to realize the entangling geometric quantum computation with a perfect fidelity.

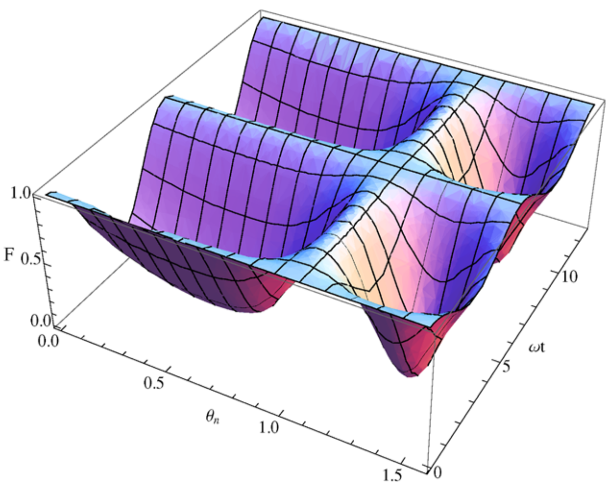

Fig. 2 Fidelity for two-particle gate as functions of θ_n and ωt, where $c_1 = \cos^2(\pi/6)$, $c_2 = \sin(\pi/6)\cos(\pi/6)$, $c_3 = \sin(\pi/6)\cos(\pi/6)$, $c_4 = \sin^2(\pi/6)$ or $c_1 = \sin^2(\pi/6)$, $c_2 = -\sin(\pi/6)\cos(\pi/6)$, $c_3 = -\sin(\pi/6)\cos(\pi/6)$, $c_4 = \cos^2(\pi/6)$. The result show that, when $\theta_n = \pi/3$ or $\omega t = 2n\pi (n = 0, 1, 2, \ldots)$ respectively, the two-particle fidelity F_2 is equal to one

8 The relation between fidelity and parameters of electric circuit

Quantum gates have the advantage that they can be realized between non-nearest qubits, possibly spatially separated by several millimeters. In addition to being interesting from a fundamental point of view, this is highly advantageous in reducing the complexity of multiqubit algorithms. Moreover, it also helps in reducing the error threshold required for reaching fault-tolerant quantum computation.

Superconducting circuits based on Josephson junctions are currently the most experimentally advanced solid state qubits. Therefore, it is interesting to analyze the fidelities for the single and two-particle gates by the parameters of electric circuit, where the typical values are taken, such as $\omega_F = 10^{10}$ Hz, $c_J = 10^{-15}$ F, $C_F = 10^{-11}$ F, $c_g/c_J = 0.1$, $E_J = 100$ mK.

The fidelities as functions of both the parameter ω of electric circuit and the evolving time t are shown at Fig. 3 for the geometric single-particle gate and Fig. 4 for the entangling geometric two-particle gate. We find that for the many regions, the fidelity has maximum

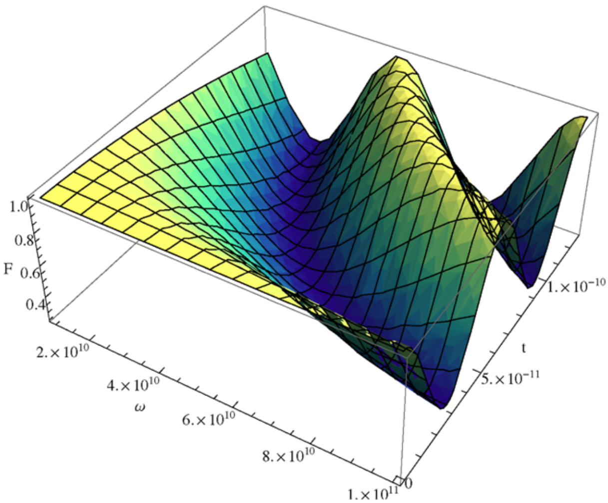

Fig. 3 Fidelity for single-particle gate as functions of ω and t, where $c_1 = \cos(\pi/3)$, $c_2 = \sin(\pi/3)$. The result shows that the fidelity oscillates with ω and t and its maximum values depend on both ω and t

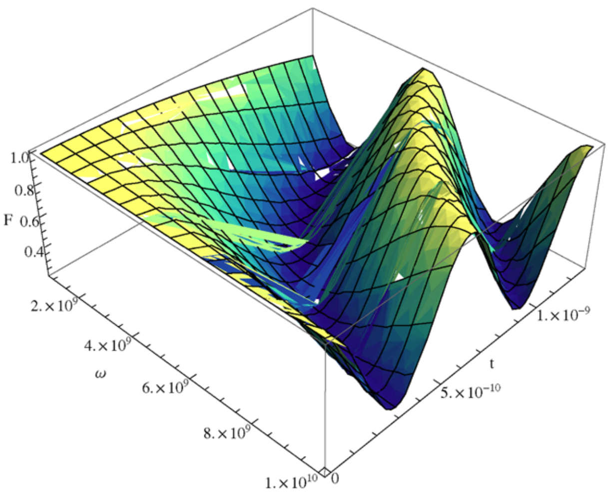

Fig. 4 Fidelity for two-particle gate as functions of ω and t, where $c_1 = \cos^2(\pi/6)$, $c_2 = \sin(\pi/6)\cos(\pi/6)$, $c_3 = \sin(\pi/6)\cos(\pi/6)$, $c_4 = \sin^2(\pi/6)$ or $c_1 = \sin^2(\pi/6)$, $c_2 = -\sin(\pi/6)\cos(\pi/6)$, $c_3 = -\sin(\pi/6)\cos(\pi/6)$, $c_4 = \cos^2(\pi/6)$ or $c_1 = -\frac{\sqrt{2}}{2}\sin(\pi/3)$, $c_2 = \frac{\sqrt{2}}{2}\cos(\pi/3)$, $c_3 = \frac{\sqrt{2}}{2}\cos(\pi/3)$, $c_4 = \frac{\sqrt{2}}{2}\sin(\pi/3)$ or $c_1 = 0$, $c_2 = \frac{\sqrt{2}}{2}$, $c_3 = -\frac{\sqrt{2}}{2}$, $c_4 = 0$

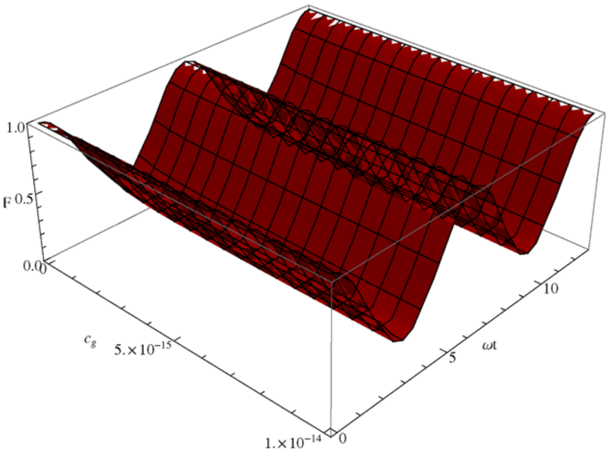

Fig. 5 Fidelity for single-particle gate as functions of c_g and ωt, where $c_1 = \cos(\pi/3)$, $c_2 = \sin(\pi/3)$

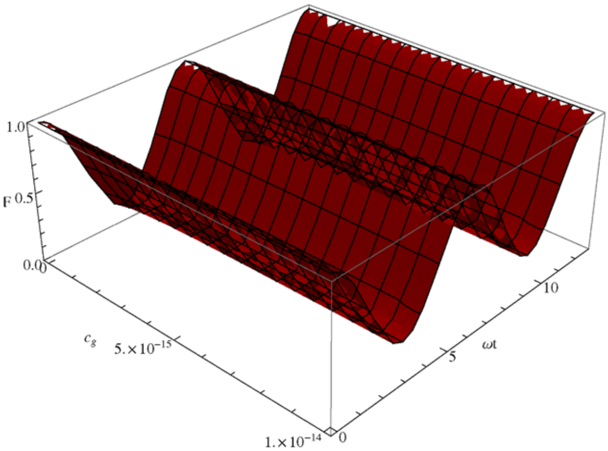

Fig. 6 Fidelity for two-particle gate as functions of c_g and ωt, where $c_1 = \cos^2(\pi/6)$, $c_2 = \sin(\pi/6)\cos(\pi/6)$, $c_3 = \sin(\pi/6)\cos(\pi/6)$, $c_4 = \sin^2(\pi/6)$ or $c_1 = \sin^2(\pi/6)$, $c_2 = -\sin(\pi/6)\cos(\pi/6)$, $c_3 = -\sin(\pi/6)\cos(\pi/6)$, $c_4 = \cos^2(\pi/6)$ or $c_1 = -\frac{\sqrt{2}}{2}\sin(\pi/3)$, $c_2 = \frac{\sqrt{2}}{2}\cos(\pi/3)$, $c_3 = \frac{\sqrt{2}}{2}\cos(\pi/3)$, $c_4 = \frac{\sqrt{2}}{2}\sin(\pi/3)$ or $c_1 = 0$, $c_2 = \frac{\sqrt{2}}{2}$, $c_3 = -\frac{\sqrt{2}}{2}$, $c_4 = 0$

values that depend on both the ω and t, which imply that the high fidelity may be obtained by choosing both the ω and t.

Differently from the ω, the affection of c_g to the fidelity is very small for the geometric single-particle gate (see Fig. 5) as well as for entangling geometric two-particle gate (see Fig. 6), where the fidelities oscillate only in terms of ωt.

9 Geometric Phases as a Degree Freedom of Fidelity

Geometric phases are important in both a fundamental point of physical view and their applications [27]. On the one hand, the physical system retains a memory of its evolution in terms of the geometric phase. On the other hand, the geometric quantum computation is a scheme intrinsically fault-tolerant and therefore resilient to certain types of computational errors. Therefore, it is very interesting in studying the relations between the fidelity and the geometric phase.

Substituting (15) into (33),we find that the single-particle fidelity may be expressed in terms of the difference $\gamma = \gamma^g_{n+} - \gamma^g_{n-}$ of geometric phases, i.e.,

$$\begin{aligned} F_1(\gamma, t) = \frac{1}{4}\bigg| & c_1c_1^*\left(1-\frac{\gamma}{2\pi}\right) + c_2c_2^*\left(1+\frac{\gamma}{2\pi}\right) - c_1c_2^*e^{i\gamma}\sqrt{1-\frac{\gamma^2}{4\pi^2}} \\ & - c_2c_1^*e^{-i\gamma}\sqrt{1-\frac{\gamma^2}{4\pi^2}} + e^{i\omega t}\left(c_1c_1^*\left(1+\frac{\gamma}{2\pi}\right) + c_2c_2^*\left(1-\frac{\gamma}{2\pi}\right)\right. \\ & \left. + c_1c_2^*e^{i\gamma}\sqrt{1-\frac{\gamma^2}{4\pi^2}} + c_2c_1^*e^{-i\gamma}\sqrt{1-\frac{\gamma^2}{4\pi^2}}\right)\bigg|^2, \end{aligned} \tag{35}$$

which means that the performance of quantum information may be used by geometric phase shifts.

Similarly, the two-particle fidelity may be rewritten by

$$\begin{aligned} F_2(\gamma, t) = \frac{1}{4}\bigg| & c_1c_1^*\left(1-\frac{\gamma}{2\pi}\right) + c_3c_3^*\left(1+\frac{\gamma}{2\pi}\right) - c_1c_3^*e^{i\gamma}\sqrt{1-\frac{\gamma^2}{4\pi^2}} \\ & - c_3c_1^*e^{-i\gamma}\sqrt{1-\frac{\gamma^2}{4\pi^2}} + e^{i\omega t}\left(c_1c_1^*\left(1+\frac{\gamma}{2\pi}\right) + c_3c_3^*\left(1-\frac{\gamma}{2\pi}\right)\right. \\ & \left. + c_1c_3^*e^{i\gamma}\sqrt{1-\frac{\gamma^2}{4\pi^2}} + c_3c_1^*e^{-i\gamma}\sqrt{1-\frac{\gamma^2}{4\pi^2}}\right) \\ & + c_2c_2^*\left(1+\frac{\gamma}{2\pi}\right) + c_4c_4^*\left(1+\frac{\gamma}{2\pi}\right) - c_2c_4^*e^{i\gamma}\sqrt{1-\frac{\gamma^2}{4\pi^2}} \\ & - c_4c_2^*e^{-i\gamma}\sqrt{1-\frac{\gamma^2}{4\pi^2}} + e^{i\omega t}\left(c_2c_2^*\left(1+\frac{\gamma}{2\pi}\right) + c_4c_4^*\left(1+\frac{\gamma}{2\pi}\right)\right. \\ & \left. + c_2c_4^*e^{i\gamma}\sqrt{1-\frac{\gamma^2}{4\pi^2}} + c_4c_2^*e^{-i\gamma}\sqrt{1-\frac{\gamma^2}{4\pi^2}}\right)\bigg|^2. \end{aligned} \tag{36}$$

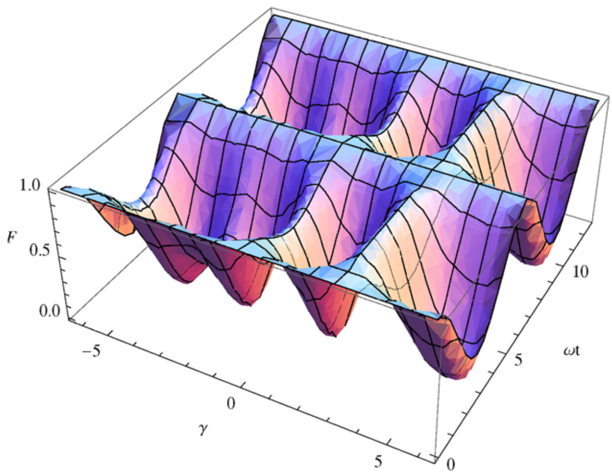

Fig. 7 Fidelity for single-particle gate as functions of γ and ωt, where $c_1 = \cos(\pi/3)$, $c_2 = \sin(\pi/3)$. The result shows that the fidelity $F = 1$ at $\gamma = \pi$ or $\omega t = 2n\pi (n = 0, 1, 2, \ldots)$

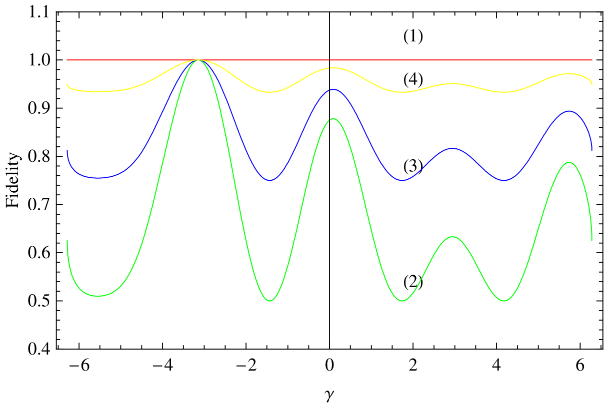

Fig. 8 Fidelity for two-particle gate as functions of γ for different ωt, (1) $\omega t = 2\pi$, (2) $\omega t = 3\pi/2$, (3) $\omega t = \pi/3$, (4) $\omega t = \pi/6$ with the initial conditions $c_1 = \cos^2(\pi/6)$, $c_2 = \sin(\pi/6)\cos(\pi/6)$, $c_3 = \sin(\pi/6)\cos(\pi/6)$, $c_4 = \sin^2(\pi/6)$ or $c_1 = \sin^2(\pi/6)$, $c_2 = -\sin(\pi/6)\cos(\pi/6)$, $c_3 = -\sin(\pi/6)\cos(\pi/6)$, $c_4 = \cos^2(\pi/6)$

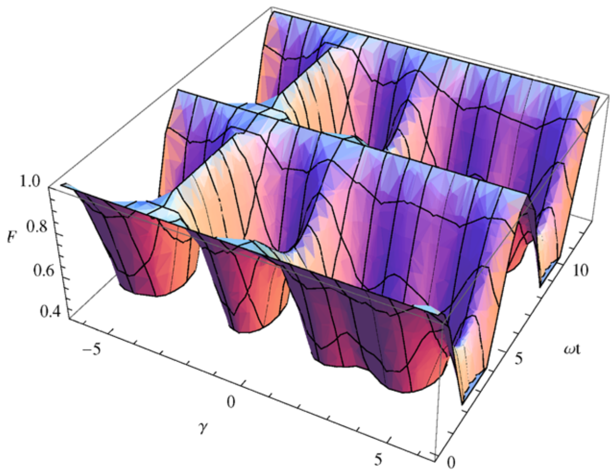

Fig. 9 Fidelity for two-particle gate as functions of γ and ωt, where $c_1 = \cos^2(\pi/6)$, $c_2 = \sin(\pi/6)\cos(\pi/6)$, $c_3 = \sin(\pi/6)\cos(\pi/6)$, $c_4 = \sin^2(\pi/6)$ or $c_1 = \sin^2(\pi/6)$, $c_2 = -\sin(\pi/6)\cos(\pi/6)$, $c_3 = -\sin(\pi/6)\cos(\pi/6)$, $c_4 = \cos^2(\pi/6)$

From (35) and (36), we see that the geometric phase may be a basic degree of freedom. Therefore $F_1(\gamma, t)$ and $F_2(\gamma, t)$ are called as geometric fidelity for the single-particle gate and two-particle gate respectively. The geometric fidelities as functions of γ and ωt are shown at Fig. 7 for the single-particle gate and at Figs. 8, 9, and 10 for the entangling two-particle gate.

From Fig. 7, we see that the geometric fidelity for the single-particle gate oscillates in terms of two directions of the difference of geometric phase γ and the evolving time ωt. Moreover, the state of single-particle system may be perfectly preserved at $\gamma = \pi$ and $\omega t = 2n\pi (n = 1, 2, \ldots)$, respectively.

It is noted that $\omega t = 2n\pi$ are the cyclicities of single-particle evolution. It is known that the nonadiabatic evolutions result in errors that typically destroy cyclicity and thereby lead to evolutions for which the conventional theory of the nonadiabatic geometric phase fail to apply. Therefore, the results at Fig. 7 propose that the gate should be manipulated by the difference of geometric phase $\gamma = \pi$ in order perfectly to preserve the message. Such scheme may be useful in the design of geometric quantum gates due to an enhanced flexibility in choice of evolutions. It also avoids the problems associated with types of errors that do not preserve cyclicity and therefore may be conceptually useful in that it makes it possible to analyze the fault tolerance associated to such errors.

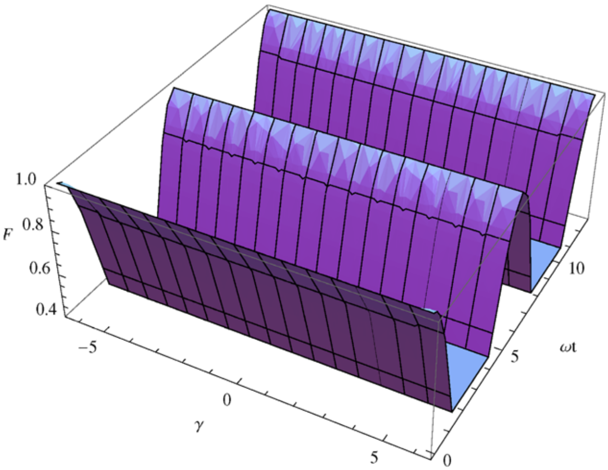

Fig. 10 Fidelity for two-particle gate as functions of γ and ωt, where $c_1 = -\frac{\sqrt{2}}{2}\sin(\pi/3)$, $c_2 = \frac{\sqrt{2}}{2}\cos(\pi/3)$, $c_3 = \frac{\sqrt{2}}{2}\cos(\pi/3)$, $c_4 = \frac{\sqrt{2}}{2}\sin(\pi/3)$ or $c_1 = 0$, $c_2 = \frac{\sqrt{2}}{2}$, $c_3 = -\frac{\sqrt{2}}{2}$, $c_4 = 0$

A similar situation is for the entangling geometric two-particle gate. From Figs. 8–10, the high fidelity may be obtained in terms of the difference of geometric phases. In addition, the geometric fidelity depend also on the initial conditions as shown at Figs. 9 and 10.

10 Conclusions

In summary, the implement of geometric quantum gates are investigated for the single-particle and two-particle system in the electric circuit. We find that the difference of geometric phase may be regarded as a basic degree of freedom of the fidelities of geometric single-particle gate as well as entangling geometric two-particle gate. by controlling the difference of geometric phase, furthermore, one may obtain a perfect fidelity that equals to one, which is not relative to the evolving time. Therefore, it may effectively decrease errors from the time manipulation. Especially, It also avoids the problems associated with types of errors that do not preserve cyclicity for the physical systems.

Acknowledgements This work is supported by Foundation of Science and Technology of Education Office of Jiangxi Province under Grant No. GJJ10400 and GJJ10404.

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